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LETTER TO THE EDITOR

Weak universality in inhomogeneous Ising quantum chains

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Abstract

The Ising quantum chain with arbitrary coupling distribution $\{\lambda_i\}$ leading to an anisotropic scaling is considered. The smallest gap of the chain is connected to the surface magnetization by the relation $\Lambda_1 = m_s(\{\lambda_i\})m_s(\{\lambda_i^{-1}\})$. For some aperiodic distribution $\{\lambda_i\}$, a weak universality of the critical behaviour is found.

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Quenched disorder has a profound impact on the nature of quantum phase transitions. In one-dimensional random magnets, the disorder leads to an extremely broad distribution of energy scales in the vicinity of the critical point [1]. Based on this remarkable property, Fisher has obtained exact results on the random transverse-field Ising quantum chain (RITF) using a direct space renormalization-group transformation [2]. A similar type of unbounded fluctuations appears in deterministic chains where the couplings follow an aperiodic sequence [3–6]. Such aperiodic systems scale essentially anisotropically in a similar fashion [7] as the random case where the length and time scales are related as $L \sim (\ln t)^{1/\omega}$ implying an infinite anisotropy exponent z, since by the definition of z, $t \sim L^z$. In the random case, outside the critical point in the so-called Griffiths phase [8], the anisotropy exponent of the RITF is finite and depends on the quantum control parameter [2]. The same strong anisotropy was observed [9] in the marginal aperiodic ITF at criticality, for logarithmically diverging fluctuations induced by the aperiodical modulation of the couplings. We show in this letter how the anisotropy exponent can be derived in general for chains with arbitrary coupling distributions that lead to an anisotropic scaling. In the marginal aperiodic case, we recover an explicit relation and we show that a weak universality emerges for some aperiodic distributions.

The Ising quantum chain in a transverse field is defined by the Hamiltonian:

$$\mathcal{H} = -\frac{1}{2}\sum_{i=1}^{L-1} J_i \sigma_i^x \sigma_{i+1}^x - \frac{1}{2}\sum_{i=1}^{L} h_i \sigma_i^z, \tag{1}$$

where σ's are the Pauli spin operators and J_i, h_i are the inhomogeneous couplings. In marginal aperiodic systems, for which $h_i = h$ and $\lambda_i = \lambda R^{f_i}$ with $\lambda_i = J_i/h_i$ and $f_i = 0, 1$ generating the aperiodicity, an anisotropic scaling was found [9, 10]. For such systems, the smallest excitations of the Hamiltonian scale at the bulk critical point as $\Lambda \sim L^{-z}$ with the size L of the chain. In the marginal aperiodic case, Berche [9, 10] showed numerically that the anisotropy exponent z is continuously varying with the control parameter R and conjectured the relation $z(R) \geqslant 1$, where

$$z(R) = x_{m_s}(R) + x_{m_s}(R^{-1}), \tag{2}$$

where $x_{m_s}(R)$ is the magnetic exponent associated with $m_s = \langle \sigma_1^x \rangle$.[1] The observed symmetry in the exchange $R \leftrightarrow 1/R$ in (2) was demonstrated in [11] for aperiodic systems generated by inflation rules, using a generalization of an exact renormalization-group method introduced first in [12] and applied to several aperiodic systems in [13]. We show here that this equation comes from a relation, valid for any distribution of couplings leading to anisotropic scaling, that relies on the first gap Λ_1 to the surface magnetization.

Using a Jordan–Wigner transformation [14], the Hamiltonian (1) can be rewritten in a quadratic form in fermion operators. It is then diagonalized by a canonical transformation and reads

$$\mathcal{H} = \sum_{q=1}^{L} \Lambda_q \left(\eta_q^\dagger \eta_q - \frac{1}{2} \right), \tag{3}$$

where $\eta_q^\dagger$ and η_q are the fermionic creation and annihilation operators. The one fermion excitations Λ_q satisfy the following set of equations:

$$\begin{aligned} \Lambda_q \Psi_q(i) &= -h_i \Phi_q(i) - J_i \Phi_q(i+1) \\ \Lambda_q \Phi_q(i) &= -J_{i-1} \Psi_q(i-1) - h_i \Psi_q(i) \end{aligned} \tag{4}$$

with the free boundary condition $J_0 = J_L = 0$. The vectors Φ and Ψ are related to the coefficients of the canonical transformation and enter into the expressions of physical quantities. For example, the surface magnetization, $m_s = \langle \sigma_1^x \rangle$, is simply given by the first component of Φ_1 associated with the smallest excitation of the chain, Λ_1.

Let us consider now a distribution of the couplings which leads to anisotropic scaling with a dynamical exponent $z > 1$, as it is the case for bulk marginal aperiodic modulation of the couplings [9–13]. Then, the bottom spectrum of the critical Hamiltonian scales as $\Lambda_q \sim L^{-z}$ in a finite size system. According to [13], the asymptotic size dependence of $\Lambda_1(L)$ is given by the expressions

$$\Lambda_1(L) \simeq (-1)^L \frac{\Psi_1(1)}{\Phi_1(L)} \prod_{i=1}^{L-1} \lambda_i^{-1} \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_{L-j}^{-2} \right]^{-1} \tag{5}$$

$$\Lambda_1(L) \simeq (-1)^L \frac{\Phi_1(L)}{\Psi_1(1)} \prod_{i=1}^{L-1} \lambda_i^{-1} \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_{j}^{-2} \right]^{-1}, \tag{6}$$

which are valid at the critical point and in the ordered phase. Noting in (5) that

$$\prod_{i=1}^{L-1} \lambda_i^{-1} \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_{L-j}^{-2} \right]^{-1} = \prod_{i=1}^{L-1} \lambda_i \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_{j}^{2} \right]^{-1}, \tag{7}$$

[1] One may note that relation (2) holds for the homogeneous system, $R = 1$, with $x_{m_s} = 1/2$ and $z = 1$.

equation (5) becomes

$$\Lambda_1(L) \simeq (-1)^L \frac{\Psi_1(1)}{\Phi_1(L)} \prod_{i=1}^{L-1} \lambda_i \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_j^2 \right]^{-1}. \tag{8}$$

Now multiplying both sides of equation (8) with (6) leads to

$$\Lambda_1(L) \simeq \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_j^{-2} \right]^{-1/2} \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_j^2 \right]^{-1/2} \tag{9}$$

which is symmetric under the exchange $\lambda \leftrightarrow 1/\lambda$. One recognizes in this expression the surface magnetization [15] $m_s(L, \{\lambda_i\}) = \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_j^{-2}\right]^{-1/2}$ of the quantum chain, so that one finally obtains

$$\Lambda_1(L) \simeq m_s(L, \{\lambda_i\}) m_s\left(L, \left\{\lambda_i^{-1}\right\}\right). \tag{10}$$

This relation connects a bulk quantity, Λ_1, with surface quantities, namely $m_s(L, \{\lambda_i\})$ and the surface magnetization of the dual chain, $m_s\left(L, \left\{\lambda_i^{-1}\right\}\right)$.

Consider now a deterministic distribution of the chain couplings, $\{\lambda_i\}$, with $h_i = h$ and $\lambda_i = JR^{f_i}/h$ with f_i following some sequence of 0 and 1. The critical coupling λ_c follows from the relation $\lim_{L\to\infty} \prod_{k=1}^{L} (J_k/h_k)^{1/L} = 1$ [16] and gives $\lambda_c = R^{-\rho_\infty}$ with ρ_∞ the asymptotic density of modified couplings λR. The modulation of the couplings introduces a perturbation which can be either relevant, marginal or irrelevant. For the Ising quantum chain, the fluctuations around the average coupling $\bar{\lambda}$ at a length scale L

$$\Delta(L) = \sum_{k=1}^{L} (\lambda_k - \bar{\lambda}) \sim L^\omega \tag{11}$$

govern the relevance of the perturbation [5, 6]. ω is a wandering exponent depending on the distribution of couplings. If $\omega < 0$, the fluctuations are bounded and the system is in the Onsager universality class. On the other hand for $\omega > 0$, the fluctuations are unbounded and one has to distinguish two different situations. The evaluation of the surface magnetization is related to the sum $\sum_{j=1}^{L} \lambda^{-2j} R^{-2n_j}$, where n_j is the number of modified couplings, λR, at size j. At the critical point, using (12) the sum can be rewritten asymptotically as $\sum_{j=1}^{L} R^{-2Bj^\omega}$. Now assume that the coefficient B is positive (if not the roles of $R > 1$ and $R < 1$ are reversed in the following discussion). For $R > 1$, the previous sum is absolutely convergent for $L \to \infty$ and leads to a finite surface magnetization with exponentially small corrections in a finite size system. On the other hand for $R < 1$, the sum is diverging exponentially and the surface magnetization is governed by the dominant term $\exp(-2B \ln RL^\omega)$, so that

$$m_s(L, R) \sim \exp(-A(R)L^\omega), \tag{12}$$

with $A(R) > 0$. So from equation (11), in both cases ($R > 1$ or $R < 1$) the first gap Λ_1 will show an essential singularity corresponding to $z = \infty$:

$$\Lambda_1(R, L) \sim \exp(-\tilde{A}(R)L^\omega), \tag{13}$$

with $\tilde{A}(R) = A(R)$ for $R < 1$ and $\tilde{A}(R) = A(1/R)$ for $R > 1$ since from (10) $\Lambda_1(R) = \Lambda_1(1/R)$.

In the marginal case, corresponding to $\omega = 0$ and a logarithmic divergence of the fluctuations, $n_j \simeq \rho_\infty j + C \ln j$ where C is some constant, it can be shown that the surface magnetization scales at the critical point as

$$m_s(L, \{\lambda_i\}_c) \sim L^{-x_{m_s}(R)} \tag{14}$$

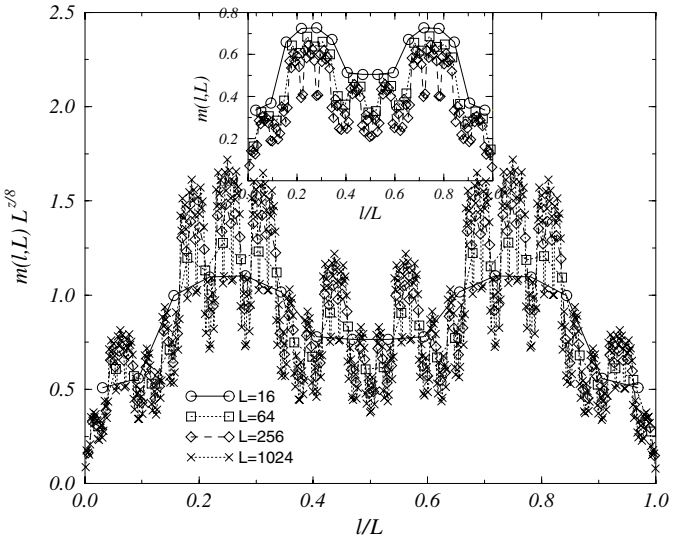

Figure 1. Rescaled magnetization profiles of the period-doubling chain with $r = 5$. The inset gives the corresponding magnetization profile.

with an exponent $x_{m_s}(R)$ varying continuously with the control parameter R. In fact, the sum can be evaluated at the critical point using $n_j \simeq \rho_\infty j + C \ln j$. One obtains $\sum_{j=1}^{L} \lambda_c^{-2j} R^{-2n_j} \simeq \sum_{j=1}^{L} R^{-2C \ln j} \sim \int^L \mathrm{d}x\, x^{-2C \ln R} \sim L^{1-2C \ln R}$. This expression is only valid in the weak perturbation regime for $R \simeq 1$, that is in first order in $\ln R$. For a stronger regime, one has to retain higher terms in the n_j expression. At this order, the surface magnetic exponent is $x_{m_s}(R) \simeq 1/2 - C \ln R$. One can remark that for a sequence like the period-doubling one [9, 10], $x_{m_s}(R) = x_{m_s}(1/R)$ which implies $C = 0$ and then the former calculation gives $x_{m_s}(R) = 1/2 + \mathcal{O}(\ln^2 R)$. Finally, from (10) and (14) one obtains relation (2). The anisotropy exponent z is then given by one surface magnetic exponent x_{m_s} which is a function of the perturbation strength. The symmetry $R \leftrightarrow 1/R$ of z is due to the self-duality of the Ising quantum chain which implies for all bulk quantities the relation $Q(\{\lambda_i\}) = Q\left(\left\{\lambda_i^{-1}\right\}\right)$.

For a symmetric distribution of couplings with respect to the centre of the chain, leading to $x_{m_s}(R) = x_{m_s}(1/R)$ (see the period-doubling case [9, 10]), one observes a weak universality. Indeed, the bottom of the spectrum scales anisotropically as $\Lambda \sim L^{-z} \sim \xi_\perp^{-z} \sim t^{z\nu}$ where t measures the deviation from the critical point and $\nu = 1$ is the exponent of the longitudinal correlation length $\xi_\perp$. So that from (10)

$$m_s(t) \sim (t^z)^{1/2}. \tag{15}$$

From anisotropic scaling, one obtains for the critical dimension of the surface energy density e_s the scaling relation $x_{e_s} = z + 2x_{m_s}$ [10]. Using the symmetry of x_{m_s}, one has $z = 2x_{m_s}$ and then

$$e_s \sim (t^z)^2. \tag{16}$$

We see that we recover the homogeneous surface exponents $x_{m_s} = 1/2, x_{e_s} = 2$ when the deviation from the critical point is measured by $t^z \sim \Lambda_1$.

The same weak universality seems to hold for the bulk quantities. In fact, it was shown in [10] that the bulk energy density scales as $e \sim L^{-z}$ for marginal aperiodic modulation of

the couplings. So again the pure energy density exponent $x_e = 1$ is recovered. Here, we have investigated the behaviour of the mean bulk magnetization $m_b = 1/L \sum_i m(i)$ for the period-doubling sequence using finite size scaling analysis. The magnetization is evaluated at the bulk critical point for sizes up to $L = 1024$. Numerically, the profiles are well rescaled on the same mean curve with an exponent $x_{m_b}(R) = z(R)/8$, confirming the weak universality scenario. One may mention that as the size increases the profiles are more and more decorated with a growing fluctuation amplitude. This suggests that the finite size behaviour of the mean critical magnetization is given by

$$m_b \sim L^{-z/8} \sim (L^{-z})^{1/8}. \tag{17}$$

On the basis of the numerical data, the magnetization profile is compatible with the form

$$m(l, L) = L^{-z/8} \, |\sin(\pi l/L)|^{x_{m_s} - x_{m_b}} \, [A + G(l/L)], \tag{18}$$

where A is a constant and $G(x)$ is a kind of fractal Weierstrass function with zero mean value whose Fourier momentum is given by the period-doubling cascade. The sine term is very general for the profiles of the Ising quantum chains and is related to the geometry of the system. This can be demonstrated explicitly for the pure case [17] and was numerically obtained for random Ising systems [18]. The only difference here is that we have not only a pure constant in the front of it but in addition a fractal function of zero mean which controls the local fluctuations, due to the aperiodic distribution, around some average environment.

In conclusion, the weak universality observed in these systems implies that the knowledge of the anisotropy exponent z, together with the universality class of the pure fixed point, is sufficient to determine the critical behaviour of the system, that is only one new exponent is needed.

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